Contents

1 Relations

1.1 Exists and Forall Quotients

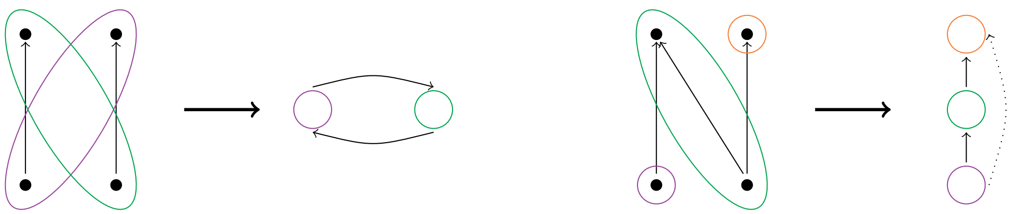
Given a poset P and an equivalence relation $\sim$, we might attempt to define an order on the quotient in one of two ways:

1.1.1 ∃-Quotient

$\forall X, Y \in P$: $X \leq_{\sim,\exists} Y \iff \exists x \in X, y \in Y : x \leq y$

This relation guarantees that the projection is order-preserving. Indeed any relation on the quotient for which the projection is order-preserving necessarily extends this relation. However the relation may fail to be a partial order.

Figure 1: Violations of poset axioms in quotients



(a) A violation of antisymmetry

(b) A violation of transitivity

1.1.2 ∀-Quotient

$\forall X, Y \in P$: $X \leq'_{\sim,\forall} Y \iff \forall x \in X, y \in Y : x \leq y$

In general the projection map will not be order-preserving onto this relation. However, we can guarantee the relation will be a partial ordering, with a slight modification to ensure reflexivity:

$\forall X, Y \in P$: $X \leq_{\sim,\forall} Y \iff \forall x \in X, y \in Y : x \leq y \vee X = Y$

Lemma 1.1. *$\leq_{\sim,\forall}$ is a partial order.*

Proof.

- Reflexivity: By construction.
- Antisymmetry: Let $X, Y \in P/\sim$ with $X \leq_{\sim,\forall} Y$ and $Y \leq_{\sim,\forall} X$. Suppose for contradiction $X \neq Y$. Then by definition of $\leq_{\sim,\forall}$, for all $x \in X, y \in Y : x \leq y$ and $y \leq x$; and $\leq$ is antisymmetric, so $x = y$. Since $X = [x]_\sim, Y = [y]_\sim$, this implies $X = Y$, a contradiction.

- Transitivity: Let $X, Y, Z \in P/\sim$ with $X \leq_{\sim,\forall} Y$ and $Y \leq_{\sim,\forall} Z$. If $X = Y$ or $Y = Z$ then trivially $X \leq_{\sim,\forall} Z$, so suppose $X \neq Y \neq Z$. Then by definition of $\leq_{\sim,\forall}$, for all $x \in X, y \in Y, z \in Z : x \leq y$ and $y \leq z$; and $\leq$ is transitive, so $x \leq z$. This holds for all $x \in X, z \in Z$, so $X \leq_{\sim,\forall} Z$.

$\square$

We see that violations of the poset axioms may occur in $\leq_{\sim,\exists}$ when different equivalence class representatives are chosen for the comparisons. Similarly $\leq_{\sim,\forall}$ fails to preserve the order under projection when not all members of a pair of equivalence classes have the same relationship. That is, failures occur when comparing by representatives is not well-defined. We turn our attention to equivalence relations where this comaprison *is* well-defined.

1.2 Reduction relations

Definition 1.2. Given $(P, \leq)$ a poset, an equivalence relation $\sim$ on P is a **reduction relation** if $\forall x_1, x_2, y_1, y_2 \in P$

$$\begin{aligned} & x_2 \sim x_1 \not\sim y_1 \sim y_2 \\ \implies & x_1 < y_1 \iff x_2 < y_2 \end{aligned} \tag{1.2.1}$$

This is equivalent to saying the relation $<_\sim$ on $P/\sim$ given by

$$[x]_\sim <_\sim [y]_\sim \iff x < y \text{ and } [x]_\sim \neq [y]_\sim \tag{1.2.2}$$

is well-defined; we call the corresponding non-strict relation $\leq_\sim$ the **quotient order**.

Remark. We explicitly do not compare a class to itself by representative since this necessitates all members of an equivalence class to be less than each other, violating antisymmetry and rendering the concept useless for partial orders. We instead declare reflexivity by fiat.

Theorem 1.3. *Given $(P, \leq)$ a poset and $\sim$ an equivalence relation on P, $\sim$ is a reduction relation iff $\leq_{\sim,\exists} = \leq_{\sim,\forall}$.*

Proof. Suppose $\leq_{\sim,\exists} = \leq_{\sim,\forall}$. Let $x_2 \sim x_1 \not\sim y_1 \sim y_2$ and $x_1 \leq y_1$.
Then,

$$\begin{aligned} & [x_1]_\sim \leq_{\sim,\exists} [y_1]_\sim \\ \implies & [x_1]_\sim \leq_{\sim,\forall} [y_1]_\sim \\ \text{Since } & x_2 \in [x_1]_\sim, y_2 \in [y_1]_\sim \\ \implies & x_2 \leq y_2 \end{aligned}$$

So $\sim$ is a reduction relation.

Suppose $\sim$ is a reduction relation. $\leq_{\sim,\forall} \subseteq \leq_{\sim,\exists}$ so it suffices to show $\leq_{\sim,\exists} \subseteq \leq_{\sim,\forall}$. Let $X, Y \in P/\sim$ with $X \leq_{\sim,\exists} Y$. If $X = Y$ then $X \leq_{\sim,\forall} Y$. Otherwise $\exists x \in X, y \in Y$ with $x \leq y$. $\forall x' \in X, y' \in Y$ we have $x \sim x'$ and $y \sim y'$ and since $\sim$ is a reduction relation and $X \neq Y$, $x \leq y \implies x' \leq y'$. Thus $X \leq_{\sim,\forall} Y$ and so $\leq_{\sim,\exists} \subseteq \leq_{\sim,\forall}$. $\square$

Corollary 1.4. *For a reduction relation on a poset, the quotient order is a partial ordering and the projection map is order-preserving.*

Proof. This follows immediately from 1.3 and the observations about $\leq_{\sim,\forall}$ and $\leq_{\sim,\exists}$. $\square$

Theorem 1.5. *Let P be a poset and $\sim$ a reduction relation on P. If R is a poset, $f : P \to R$ is order-preserving and $\sim \subseteq \ker f$, then the factor map $\bar{f} : P/\sim \to R$ satisfying $\bar{f}([x]_\sim) = f(x)$ is order-preserving.*

Proof. Let $[x], [y] \in P/\sim$ with $[x] \leq [y]$.

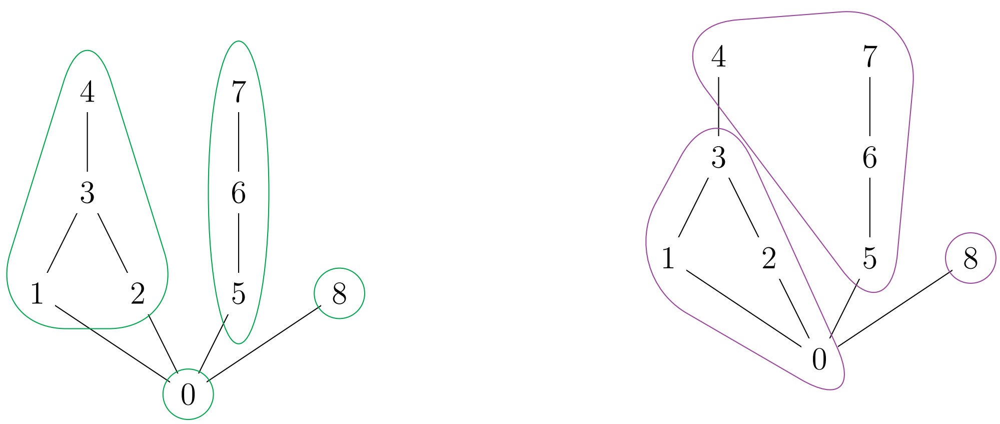
If $[x] = [y]$ then $\bar{f}([x]) = \bar{f}([y])$.

If $[x] < [y]$ then from definition (1.2.2), $x < y$ and since f is order-preserving

$$\bar{f}([x]) = f(x) \leq f(y) = \bar{f}([y])$$

So in all cases $\bar{f}([x]) \leq \bar{f}([y])$ $\square$

Figure 2: Reduction relation examples



(a) A reduction relation

(b) Not a reduction relation

2 Compositions and Reductions

2.1 Compositions

Definition 2.1. Given a collection of pairwise disjoint posets $\{(P_\alpha, \leq_\alpha)\}_{\alpha\in I}$, and a partial order $\leq$ on $\{P_\alpha\}_{\alpha\in I}$, the **composition** $\leq [\leq_\alpha]_{\alpha\in I}$ is the relation $\preceq$ on $\bigcup_{\alpha\in I} P_\alpha$ given by

$$\forall \alpha \in I, x, y \in P_\alpha : x \preceq y \iff x \leq_\alpha y \tag{2.1.1}$$

$$\forall \alpha \neq \beta \in I, x \in P_\alpha, y \in P_\beta : x \prec y \iff P_\alpha < P_\beta \tag{2.1.2}$$

That is, compare elements in the same block using that block's order, and elements from different blocks using the ordering on the blocks.

Theorem 2.2. *If $\{(P_\alpha, \leq_\alpha)\}_{\alpha\in I}$ is a collection of disjoint posets and $\leq$ is a partial order on $\{P_\alpha\}_{\alpha\in I}$, $\leq [\leq_\alpha]_{\alpha\in I}$ is a partial order on $\bigcup_{\alpha\in I} P_\alpha$.*

Proof. Let $P := \bigcup_{\alpha\in I} P_\alpha$ and $\preceq := \leq [\leq_\alpha]_{\alpha\in I}$.

Reflexivity: Let $x \in P$. Then $x \in P_\alpha$ for some $\alpha \in I$. $\leq_\alpha$ is reflexive, hence $x \leq_\alpha x$ and from (2.1.1), $x \preceq x$.

Antisymmetry: Let $x, y \in P$ such that $x \preceq y$ and $y \preceq x$ and let $\alpha, \beta \in I$ with $x \in P_\alpha, y \in P_\beta$. If $\alpha \neq \beta$, then $x \neq y$ so by (2.1.2) $P_\alpha < P_\beta$ and $P_\beta < P_\alpha$; but $<$ is antisymmetric so this is a contradiction and so $\alpha = \beta$. Then by (2.1.1) $x \leq_\alpha y$ and $y \leq_\alpha x$ and $\leq_\alpha$ is antisymmetric, so $x = y$.

Transitivity: Let $x, y, z \in P$ such that $x \preceq y$ and $y \preceq z$, and let $\alpha, \beta, \gamma \in I$ with $x \in P_\alpha, y \in P_\beta, z \in P_\gamma$.

If $\alpha = \beta = \gamma$ then by (2.1.1) $x \leq_\alpha y$ and $y \leq_\alpha z$. $\leq_\alpha$ is transitive so $x \leq_\alpha z$ and then by (2.1.1) again $x \preceq z$.

If $\alpha = \beta \neq \gamma$, then $y \neq z$ and by (2.1.2) $y \prec z \implies P_\beta < P_\gamma$ and then by (2.1.2) $x \in P_\beta, z \in P_\gamma \implies x \prec z$.

If $\alpha \neq \beta = \gamma$, then $x \neq y$ and by (2.1.2) $x \prec y \implies P_\alpha < P_\beta$ and then by (2.1.2) $x \in P_\alpha, z \in P_\beta \implies x \prec z$.

If $\alpha \neq \beta$ and $\beta \neq \gamma$, then $x \neq y$ and $y \neq z$ and by (2.1.2) $x \prec y \implies P_\alpha < P_\beta$ and $y \prec z \implies P_\beta < P_\gamma$. By transitivity of $<$, $P_\alpha < P_\gamma$ (in particular $P_\alpha \neq P_\gamma$) and thus by (2.1.2) $x \in P_\alpha, z \in P_\gamma \implies x \prec z$. $\square$

2.2 Reductions

Definition 2.3. A reduction is an anti-composition: a partition of a poset such that the restrictions of the order to the blocks can be composed to reobtain the original order.

That is $(R, \leq_R)$ is a **reduction** of a poset $(P, \leq_P)$ if R is a partition of P, $\leq_R$ is a partial order on R and

$$\leq_P = \leq_R [\leq_P |_X]_{X \in R}$$

Note that necessarily for disjoint posets $(P_\alpha, \leq_\alpha)$, $(\{P_\alpha\}_\alpha, \leq)$ is a reduction of $(\bigcup_\alpha P_\alpha, \leq [\leq_\alpha]_\alpha)$.

Theorem 2.4. *If $(P, \leq)$ is a poset and $\sim$ is a reduction relation on P, then $(P/\sim, \leq_\sim)$ is a reduction of P.*

Proof. Let $\leq' := \leq_\sim [\leq |_X]_{X \in P/\sim}$. Let $x, y \in P$

If $y \in [x]_\sim$ then

$$\begin{aligned} & x \leq' y \\ \overset{(2.1.1)}{\iff} & x \leq |_{[x]} y \\ \iff & x \leq y \end{aligned}$$

If $y \notin [x]_\sim$ then $y \neq x$ and

$$\begin{aligned} & x <' y \\ \overset{(2.1.2)}{\iff} & [x] <_\sim [y] \\ \overset{(1.2.2)}{\iff} & x < y \end{aligned}$$

So $\leq = \leq'$. $\square$

Theorem 2.5. *If $(P, \leq_P)$ is a poset and $(R, \leq_R)$ is a reduction of P, then the equivalence relation $\sim_R$ corresponding to R is a reduction relation and $\leq_R = \leq_{\sim_R}$, the quotient order on $P/\sim_R$.*

Proof. Suppose $x_2 \sim_R x_1 \not\sim_R y_1 \sim_R y_2$. Then there exist $X \neq Y \in R$ with $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. By (2.3) $\leq_P = \leq_R [\leq_P |_X]_{X \in R}$, and $x_1 \neq y_1, x_2 \neq y_2$, so

$$\begin{aligned} & x_1 \leq_P y_1 \\ \iff & x_1 (\leq_R [\leq |_X]_{X \in R}) y_1 \\ \overset{(2.1.2)}{\iff} & X <_R Y \\ \overset{(2.1.2)}{\iff} & x_2 (\leq_R [\leq |_X]_{X \in R}) y_2 \\ \iff & x_2 \leq_P y_2 \end{aligned}$$

So $\sim_R$ is a reduction relation.

$\forall X, Y \in R$, there exist $x, y \in P$ with $X = [x]_{\sim_R}$ and $Y = [y]_{\sim_R}$.

$$\begin{aligned} & X <_R Y \\ \overset{(2.1.2)}{\Longleftrightarrow} & x < y \\ \overset{(1.2.2)}{\Longleftrightarrow} & X = [x]_{\sim_R} <_{\sim_R} [y]_{\sim_R} = Y \end{aligned}$$

So $\leq_R = \leq_{\sim_R}$. $\square$

Theorem 2.5 gives us that a partition R of a poset P has at most one order yielding a reduction, so we can unambiguously refer to R as a reduction of P. Further, taking theorems 2.4 and 2.5 together we see that a partition is a reduction if and only if the corresponding equivalence relation is a reduction relation.

3 Maps

Definition 3.1. Let $(P, \leq_P)$ and $(R, \leq_R)$ be posets. A function $f : P \to R$ is a **reduction map** if $\forall x, y \in P$

$$x \leq_P y \implies f(x) \leq_R f(y) \tag{3.1.1}$$

$$f(x) <_R f(y) \implies x <_P y \tag{3.1.2}$$

or equivalently

$$\begin{gathered} f(x) \neq f(y) \implies \\ x <_P y \iff f(x) <_R f(y) \end{gathered} \tag{3.1.3}$$

Remark 3.2. For $(P, \leq_P)$ and $(R, \leq_R)$ posets and $f : P \to R$ recall f is:

(3.2.1a)	**order-preserving** if	$x \leq_P y \implies f(x) \leq_R f(y)$
(3.2.1b)	**strictly order-preserving** if	$x <_P y \implies f(x) <_R f(y)$
(3.2.1c)	**order-reflecting** if	$f(x) \leq_R f(y) \implies x \leq_P y$
(3.2.1d)	**strictly order-reflecting** if	$f(x) <_R f(y) \implies x <_P y$

Thus a reduction map is an order-preserving, strictly order-reflecting map. It is straightforward to check each of these properties is closed under composition, hence reduction maps are closed under composition.

Lemma 3.3. *f is order-reflecting iff it is injective and strictly order-reflecting.*

Proof. Suppose f is order-reflecting and let $x, y \in P$. If $f(x) = f(y)$ then

$$\begin{aligned} & f(x) \leq_R f(y) \text{ and } f(y) \leq_R f(x) \\ \xRightarrow{(3.2.1c)} & x \leq_P y \text{ and } y \leq_P x \\ \implies & x = y \end{aligned}$$

So f is injective. If $f(x) <_R f(y)$ then $x \neq y$ and by (3.2.1c) $x \leq_P y$, so $x <_P y$ and f is strictly order-reflecting.

Suppose f is injective and strictly order-reflecting and let $x, y \in P$. If $f(x) \leq_R f(y)$ either:

- $f(x) = f(y)$ which by injectivity means $x = y \implies x \leq_P y$
- Or $f(x) <_R f(y)$ which by (3.2.1d) means $x <_P y \implies x \leq_P y$

So (3.2.1c) is satisfied and f is order-reflecting. $\square$

Theorem 3.4. *f is an order embedding iff it is an injective reduction map.*

Proof. By definition an order embedding is an order-reflecting, order-preserving map. By (3.3) this is equivalent to an injective, strictly order-reflecting, and order-preserving map, which is quivalent to an injective reduction map. □

Remark 3.5. We can observe that **strictly order-preserving** implies **order-preserving** and **order-reflecting** implies **strictly order-reflecting**. From (3.3) order-reflecting implies injectivity, and for an injective map, order-preserving and strictly order-preserving are equivalent, so being an embedding is also equivalent to being a strictly order-preserving, order-reflecting map. That is being an embedding can be seen to be taking the stronger of each property pair, while being a reduction map is taking the weaker of each property pair.

The following two theorems show that reduction relations are precisely the kernels of reduction maps.

Theorem 3.6. *Let $(P, \leq_P)$ and $(R, \leq_R)$ be posets and let $f : P \to R$ be a reduction map. Then $\ker f$ is a reduction relation.*

Proof. Let $x_1, x_2, y_1, y_2 \in P$ such that $f(x_2) = f(x_1) \neq f(y_1) = f(y_2)$. Then if $x_1 <_P y_1$, 3.1.1 implies $f(x_1) \leq_R f(y_1)$; $f(x_1) = f(x_2)$ and $f(y_1) = f(y_2)$ imply $f(x_2) \leq_R f(y_2)$ and $f(x_2) \neq f(y_2)$ implies $f(x_2) <_R f(y_2)$ so from 3.1.2 $x_2 <_P y_2$. □

Theorem 3.7. *Let $\sim$ be a reduction relation on a poset $(P, \leq_P)$. Then the projection map $[\cdot]_\sim : P \to P/\sim$ is a reduction map.*

Proof. This follows immediately from 1.2.2 and 3.1.3. □

Sections are usually defined as right inverses to surjective functions; here we broaden that definition slightly to include arbitrary functions:

Definition 3.8. For a function $f : A \to B$, a **pseudosection** of f is a function $g : \mathrm{im}(f) \to A$ such that $fg = \mathrm{id}_{\mathrm{im}(f)}$.

Lemma 3.9. *If $f : A \to B$ is a function and $f(x) \neq f(y)$ for some $x, y \in A$, there exists a pseudosection g of f with $g(f(x)) = x$ and $g(f(y)) = y$.*

Proof. Let $g' : \mathrm{im}(f) \to A$ be an arbitrary pseudosection of f, and define $g : \mathrm{im}(f) \to A$ as

$$g(a) = \begin{cases} x & \text{for } a = f(x) \\ y & \text{for } a = f(y) \\ g'(a) & \text{otherwise} \end{cases}$$

which is well-defined when $f(x) \neq f(y)$. □

Theorem 3.10. *Let $(P, \leq_P)$ and $(R, \leq_R)$ be posets and let $f : P \to R$ be a function. TFAE:*

1. *f is a reduction map.*

2. *f is order-preserving and every pseudosection of f is order-preserving*

3. *Every pseudosection of f is an embedding*

Proof.
$1 \implies 2$) Reduction maps are order-preserving, so it suffices to show every pseudosection is order-preserving. Let g be a pseudosection of f, and suppose $X \leq_R Y$ for some $X, Y \in \operatorname{im}(f)$. If $X = Y$ then $g(X) = g(Y) \implies g(X) \leq_P g(Y)$. Otherwise $f(g(X)) = X <_R Y = f(g(Y))$ so by 3.1.2 $g(X) <_P g(Y) \implies g(X) \leq_P g(Y)$. So g is order-preserving.

$2 \implies 3$) Let g be a pseudosection of f. Suppose $g(X) \leq_P g(Y)$ for some $X, Y \in \operatorname{im}(f)$. Then since f is order-preserving, $X = f(g(X)) \leq_R f(g(Y)) = Y$. So g is order-reflecting and by assumption order-preserving, hence an embedding.

$3 \implies 1$) Order-preserving: Let $x, y \in P$ with $x \leq_P y$. If $f(x) = f(y)$ then trivially $f(x) \leq_R f(y)$. Otherwise, by (3.9) there exists a pseudosection g of f taking $f(x)$ to x and $f(y)$ to y. Then $g(f(x)) = x <_P y = g(f(y))$ and by assumption g is an embedding, so $f(x) <_R f(y)$.

Strictly order-reflecting: Let $x, y \in P$ with $f(x) <_R f(y)$ and let g be a pseudosection of f taking $f(x)$ to x and $f(y)$ to y. By assumption g is an embedding so $x = g(f(x)) <_P g(f(y)) = y$. □

In particular from 3.4 all pseudosections of a reduction map are reduction maps.

3.1 First isomorphism theorem

Theorem 3.11. *Let $(P, \leq_P)$ and $(R, \leq_R)$ be posets and let $\sim$ be a reduction relation on P. If $f : P \to R$ is a reduction map with $\sim \subseteq \ker f$, then the factor map $\bar{f}$ is a reduction map.*

Proof. Since f is a reduction map it is order-preserving, so by theorem 1.5 $\bar{f}$ is order-preserving.

Let $X, Y \in P/\sim$ with $\bar{f}(X) <_R \bar{f}(Y)$. There exist $x, y \in P$ with $X = [x], Y = [y]$ so we have $\bar{f}(X) = \bar{f}([x]) = f(x)$ and $\bar{f}(Y) = \bar{f}([y]) = f(y)$. Then $f(x) <_R f(y)$ and f is strictly order reflecting, hence $x <_P y$ and $[\cdot]$ is order-preserving so $X = [x] \leq_\sim [y] = Y$. $\bar{f}(X) <_R \bar{f}(Y) \implies \bar{f}(X) \neq \bar{f}(Y) \implies X \neq Y$, so $X <_\sim Y$ and thus $\bar{f}$ is strictly order reflecting. □

As a corrolary

Theorem 3.12 (First Isomorphism Theorem)**.**
If $(P, \leq_P)$ and $(R, \leq_R)$ are posets and $f : P \to R$ is a reduction map, then $P/\ker f \cong \operatorname{im} f$.

Proof. Letting $\sim := \ker f$ in theorem 3.11, $\bar{f} : P/\ker f \to R$ is a reduction map and must be injective, so from theorem 3.4 $\bar{f}$ is an order embedding, which is precisely an isomorphism onto $\operatorname{im} f$. □

4 Components

Definition 4.1. A non-empty subset C of a poset $(P, \leq)$ is a **component** of P if $\forall x, y \in C, z \in P \setminus C$

$$
\begin{array}{ccc} x < z & & z < x \\ \Updownarrow & \text{and} & \Updownarrow \\ y < z & & z < y \end{array} \tag{4.1.1}
$$

Notation. For x, y in a poset $(P, \leq)$ exactly one is true: $x = y$, $x < y$, $x > y$, or x and y are incomparable. We define the function $\leq^*\colon P \times P \to \{=, <, >, ||\}$ taking a pair to their relationship. (*todo: Should avoid overloading the symbols*)

The preceeding condition then becomes $\leq^*(x, z) = \leq^*(y, z)$, while the reduction relation condition (1.2.1) becomes

$$
\begin{aligned} & x_2 \sim x_1 \not\sim y_1 \sim y_2 \\ \implies & \leq^*(x_1, y_1) = \leq^*(x_2, y_2) \end{aligned} \tag{4.1.2}
$$

and the quotient order satisfies

$$
\leq^*_R([x]_R, [y]_R) = \leq^*(x, y) \text{ when } [x]_\sim \neq [y]_\sim \tag{4.1.3}
$$

Intuitively, a component is a subset that may have non-trivial local structure, but globally its points compare the same to all external points; thus it can be collapsed to a point without losing the global structure.

Theorem 4.2. *Let $(P, \leq)$ be a poset and $\sim$ an equivalence relation on P. Then $\sim$ is a reduction relation iff each equivalence class of $\sim$ is a component.*

Proof. Suppose $\sim$ is a reduction relation and let $C \in P/\sim$. Let $x, y \in C$, $z \in P \setminus C$. Then $C = [y] = [x] \neq [z] = [z]$ so by (4.1.2) $\leq^*(x, z) = \leq^*(y, z)$ and so C is a component.

Suppose that every element of $P/\sim$ is a component. Let $x_2 \sim x_1 \not\sim y_1 \sim y_2$. Since $x_1, x_2 \in [x_1]$, $y_1 \notin [x_1]$, and $[x_1] \in P/\sim$ is a component of P, from (4.1.1) $\leq^*(x_1, y_1) = \leq^*(x_2, y_1)$. Similarly, since $y_1, y_2 \in [y_1]$, $x_2 \notin [y_1]$, $\leq^*(x_2, y_1) = \leq^*(x_2, y_2)$. Taken together we have, $\leq^*(x_1, y_1) = \leq^*(x_2, y_2)$, satisfying (4.1.2). $\square$

In other words, a reduction is a partition into components.

Lemma 4.3. *If $(P, \leq)$ is a poset and $\{C_\alpha\}$ is a collection of components of P, then $\bigcap_\alpha C_\alpha$ is either empty or a component of P.*

Proof. Suppose $\bigcap_\alpha C_\alpha$ is non-empty and let $x, y \in \bigcap_\alpha C_\alpha$, $z \notin \bigcap_\alpha C_\alpha$. Then there exists some α for which $z \notin C_\alpha$. Necessarily $x, y \in C_\alpha$, and C_α is a component of P, so $\leq^*(x, z) = \leq^*(y, z)$ satisfying (4.1.1). $\square$

Lemma 4.4. *If $(P, \leq)$ is a poset and C and D are components of P with non-empty intersection, then $C \cup D$ is a component of P.*

Proof. Fix $w \in C \cap D$. Let $x, y \in C \cup D$ and $z \notin C \cup D$. x must be in either C or D, w is in both, and z is in neither so $\leq^*(x, z) = \leq^*(w, z)$. Similarly, $\leq^*(w, z) = \leq^*(y, z)$. Taken together, $\leq^*(x, z) = \leq^*(y, z)$, so $C \cup D$ is a component. $\square$

Lemma 4.5. *If $(P, \leq)$ is a poset, C is a component of P and D is a component of $(C, \leq|_C)$, then D is a component of P.*

Proof. Let $x, y \in D, z \notin D$. If $z \in C \setminus D$ then since D is a component of C,

$$\leq^*(x, z) = \leq|_C^*(x, z) = \leq|_C^*(y, z) = \leq^*(y, z)$$

Otherwise $z \in P \setminus C$ and $x, y \in C$ a component of P, so $\leq^*(x, z) = \leq^*(y, z)$. $\square$

5 Reduction Semilattice

For a partition R of a set P we state the following without proof, for all $x \in P$:

- If there exists $B \subseteq R$ with $x \in \bigcup B$, $[x]_R \in B$

Lemma 5.1. *For a poset $(P, \leq)$ and reduction R of P, the projection $\pi_R[C]$ of a component C of P is a component of R.*

Proof. Let $[x]_R, [y]_R \in \pi_R[C]$ for some $x, y \in C$, and let $[z]_R \in R \setminus \pi_R[C]$. Necessarily $z \notin C$ and C is a component, so $\leq^*(x, z) = \leq^*(y, z)$. $[x]_R \neq [z]_R \neq [y]_R$ so from (4.1.3) $\leq^*_R([x]_R, [z]_R) = \leq^*_R([y]_R, [z]_R)$. ☐

Lemma 5.2. *For a poset $(P, \leq)$ and reduction R of P, the union $\bigcup D$ of a component D of R is a component of P.*

Proof. Let $x, y \in \bigcup D$ and let $z \notin \bigcup D$. Then $[x]_R, [y]_R \in D$ and $[z]_R \notin D$ with D a component, so $\leq^*_R([x]_R, [z]_R) = \leq^*_R([y]_R, [z]_R)$ and $[x]_R \neq [z]_R \neq [y]_R$, so from (4.1.3) $\leq^*(x, z) = \leq^*(y, z)$. ☐

The operations $\pi_R[\cdot]$ and $\bigcup \cdot$ map between components of P and components of R. In general $\pi_R[\cdot]$ is not injective on components of P. However lemma (5.2) suggests we restrict our attention to the unions of components of R.

Definition 5.3. For a poset $(P, \leq)$ and reduction R of P, a subset $A \subseteq P$ is R**-compatible** if it can be written as a union of members of R, or equivalently if $\forall x \in A : [x]_R \subseteq A$.

Theorem 5.4. $\pi_R[\cdot] : \{$*R-compatible components of P*$\} \to \{$*Components of R*$\}$ *is bijective, with inverse* $\bigcup \cdot : \{$*Components of R*$\} \to \{$*R-compatible components of P*$\}$

Proof. If C is an R-compatible component of P, $\bigcup \pi_R[C] = C$: $\forall x \in C$, $[x]_R \in \pi_R[C]$ and $x \in [x]_R$ so $x \in \bigcup \pi_R[C]$. $\forall x \in \bigcup \pi_R[C]$, $[x]_R \in \pi_R[C]$ so $\exists y \in C, [y]_R = [x]_R$. C is R-compatible so $[y]_R \subseteq C$ and $x \in [y]_R$ so $x \in C$.

If D is a component of R, $\pi_R[\bigcup D] = D$: $\forall X \in D$, let $x \in X$. Then $x \in \bigcup D$ and $X = [x]_R \in \pi_R[\bigcup D]$. $\forall X \in \pi_R[\bigcup D]$, let $X = [x]_R$ for some $x \in \bigcup D$. Then $X = [x]_R \in D$. ☐

So the components of R and the R-compatible components of P can be naturally identified. This extends to an identification of the reductions of R with the reductions of P coarser than R. [1]

Theorem 5.5 (Lattice theorem)**.** *For a poset $(P, \leq)$ and reduction R of P, the set of reductions of P coarser than R and the set of reductions of R are in one-to-one correspondence under the mappings:*

$$F : \{\textit{Reductions of } P \textit{ coarser than } R\} \to \{\textit{Reductions of } R\}$$
$$F = S \mapsto \{\pi_R[C], C \in S\}$$

[1] In fact the identification of reductions can be expanded to an isomorphism of categories.

and

$$G : \{\textit{Reductions of } R\} \to \{\textit{Reductions of } P \textit{coarser than } R\}$$
$$G = T \mapsto \left\{\bigcup D, D \in T\right\}$$

Proof. We first wish to show that for a reduction S of P coarser than R and a reduction T of R, $F(S)$ is in fact a reduction of R and $G(T)$ is a reduction of P coarser than R.

For $[x]_R \in R$, $x \in [x]_S \in S$ so $[x]_R \in \pi_R[[x]_S] \in F(S)$. Hence $F(S)$ covers R. Suppose $\exists C_1, C_2 \in S$ with $\pi_R[C_1] \cap \pi_R[C_2] \neq \emptyset$. In particular there must exist $X \in \pi_R[C_1] \cap \pi_R[C_2]$ meaning $\exists x_1 \in C_1, x_2 \in C_2 : [x_1]_R = X = [x_2]_R$. Since S is coarser than R, C_1 and C_2 are both R-compatible so $X = [x_1]_R \subseteq C_1$ and $X = [x_2]_R \subseteq C_2$. Then $C_1 \cap C_2 \neq \emptyset$ but S is a partition so $C_1 = C_2$ and $\pi_R[C_1] = \pi_R[C_2]$. Thus $F(S)$ is pairwise disjoint.

So $F(S)$ forms a partition of R and from (5.1) each $\pi_R[C]$ is a component for all $C \in S$ so $F(S)$ is a reduction of R by (4.2).

Let $x \in P$. Then $x \in [x]_R$, $[x]_R \subseteq \bigcup[[x]_R]_T$ and $\bigcup[[x]_R]_T \in G(T)$ so $G(T)$ covers P. Suppose $\exists D_1, D_2 \in T$ with $\bigcup D_1 \cap \bigcup D_2 \neq \emptyset$. Then $\exists X_1 \in D_1, X_2 \in D_2$ with $X_1 \cap X_2 \neq \emptyset$. R is a partition so $X_1 = X_2$ and so $D_1 \cap D_2 \neq \emptyset$ and T is a partition so $D_1 = D_2$ and then $\bigcup D_1 = \bigcup D_2$. Hence $G(T)$ is pairwise disjoint.

So $G(T)$ forms a partition of P and for each $D \in T$, $\bigcup D$ is a component of P from (5.2) so again by (4.2), $G(T)$ is a reduction of P, and each $\bigcup D$ is R-compatible by construction, so $G(T)$ is coarser than R.

Finally applying (5.4)

$$GF(S) = \left\{\bigcup D : D \in F(S)\right\} = \left\{\bigcup \pi_R[C] : C \in S\right\} = \{C : C \in S\} = S$$

and

$$FG(T) = \{\pi_R[C] : C \in G(T)\} = \left\{\pi_R\left[\bigcup D\right] : D \in T\right\} = \{D : D \in T\} = T$$

so $F^{-1} = G$. □

From this we see the "is a reduction of" relation is essentially transitive and every poset can be considered the top of a join-semilattice of reductions; the common refinement of reductions is again a reduction, giving arbitrary joins.

Lemma 5.6. *The (arbitrary) intersection of reduction relations is a reduction relation; equivalently the common refinement of reductions is a reduction.*

Proof. Let $\{\sim_\alpha\}_\alpha$ be reduction relations on a poset P, and let $\sim := \bigcap_\alpha \sim_\alpha$. As the intersection of equivalence relations, $\sim$ is an equivalence relation; whenever $x_2 \sim x_1 \not\sim y_1 \sim y_2$, for each α, $x_2 \sim_\alpha x_1 \not\sim_\alpha y_1 \sim_\alpha y_2$ and $\sim_\alpha$ is a reduction relation so $x_1 < y_1 \iff x_2 < y_2$ and so by (1.2.1) $\sim$ is a reduction relation. □

6 Canonical

6.1 Irreducible reductions

Definition 6.1. A poset $(P, \leq)$ is **irreducible** if it admits no reductions other than the identity reduction and the trivial reduction.

Theorem 6.2. *Every finite poset admits an irreducible reduction.*

Proof. If a poset admits no irreducible reduction, we can construct an infinite sequence of non-identity reductions which transitively are each a reduction of the original, using the identification in theorem (5.5). For a finite poset, any non-identity reduction must be of strictly lesser order, so the preceeding is impossible. $\square$

Lemma 6.3. *A poset $(P, \leq)$ is irreducible if and only if it contains no proper non-singleton components.*

Proof. Suppose C is a proper non-singleton component of P. Then in the partition $\{\{x\} : x \in P \setminus C\} \cup \{C\}$, since singletons are trivially components, this forms a reduction which is non-trivial (since C is proper) and non-identity (since C is non-singleton)

Suppose R is a non-trivial, non-identity reduction of P. In particular, there must exist some non-singleton $C \in R$ (since R is non-identity) and C must be proper (since R is non-trivial) and as a member of R, C is a component of P. $\square$

We now show that if a poset P admits an irreducible reduction of order greater than 2, it is the unique irreducible reduction of P and further that it must be a reduction of all nontrivial reductions of P. Note that this is truly unique, not only up to isomorphism, as a partition of the set P.

Lemma 6.4. *Let $(P, \leq)$ be a poset, let R be a reduction of P, and C be a component of P. Then $C' := \{x \in P : [x]_R \subseteq C\}$ is a component of P, and by (5.1) $\pi_R[C']$ is a component of R.*

Proof. Let $x, y \in C', z \in P \setminus C'$. Then $\exists z' \in [z]_R \setminus C$, and since $x, y \in C' \subseteq C$ and C is a component, it follows that $\leq^*(x, z') = \leq^*(y, z')$. Necessarily $[x]_R \neq [z']_R = [z]_R$ and $[y]_R \neq [z']_R = [z]_R$, so from (4.1.2) $\leq^*(x, z) = \leq^*(x, z')$ and $\leq^*_R(y, z) = \leq^*_R(y, z')$. Taking these three equalities together, $\leq^*(x, z) = \leq^*_R(y, z)$, completing the proof. $\square$

Lemma 6.5. *Let $(P, \leq)$ be a poset and let Q be an irreducible reduction of P with $|Q| > 2$. If C is a component of P such that there exist $X_1 \neq X_2 \in Q$ with $X_1 \cap C \neq \emptyset \neq X_2 \cap C$, then $C = P$.*

Proof. Since $|Q| > 2$, $\{X_1, X_2\}$ is a proper non-singleton subset of the irreducible Q, hence it is not a component by (6.3). So there must exist some $X_3 \in Q$ distinct from X_1 and X_2 such that

$$\leq^*_Q(X_1, X_3) \neq \leq^*_Q(X_2, X_3)$$

Let $x_1 \in X_1 \cap C, x_2 \in X_2 \cap C$. Then for all $x_3 \in X_3$ since $[x_3]_Q = X_3 \neq X_1 = [x_1]_Q$ and $[x_3]_Q = X_3 \neq X_2 = [x_2]_Q$, by (4.1.3)

$$\leq^*(x_1, x_3) \neq \leq^*(x_2, x_3)$$

But then since $x_1, x_2 \in C$ which is a component, it follows that $x_3 \in C$. x_3 was arbitrary in X_3 so $X_3 \subseteq C$. Letting C' be defined as in (6.4) we thus have $X_3 \in \pi_Q[C']$.

Now $X_3 \cap C \neq \emptyset$ and $X_3 \neq X_1$, so applying the preceeding reasoning again to X_1 and X_3, there must exist some $X_4 \in Q$ distinct from X_1 and X_3 (possibly equal to X_2) with $X_4 \in \pi_Q[C']$.

By (6.4) $\pi_Q[C']$ is a component of Q and since it contains X_3 and X_4 it is non-singleton, so we must have $\pi_Q[C'] = Q$. But then $P = \bigcup Q = \bigcup \pi_Q[C'] \subseteq C$ so $C = P$. $\square$

Equivalently, any proper component C of P must be fully contained in some unique $X \in Q$, yielding the following:

Theorem 6.6. *If Q is an irreducible reduction of $(P, \leq)$ with $|Q| > 2$, and R is any reduction of P, Q is coarser than R.*

Corollary 6.7. *If Q is an irreducible reduction of $(P, \leq)$ with $|Q| > 2$, then Q is the unique irreducible reduction of P.*

6.2 Canonical reductions

In general, when a poset admits an irreducible reduction of order 2 it will not be unique. However, up to isomorphism there are only two posets of order 2, $\cdot\ \cdot$ and $\begin{smallmatrix}\bullet\\ \vert \\ \bullet\end{smallmatrix}$, and a given poset may only admit reductions isomorphic to one or the other as we show below; that is if a poset admits an irreducible reduction, it is unique up to isomorphism.

Lemma 6.8. *A poset cannot admit reductions to both $\cdot\ \cdot$ and $\begin{smallmatrix}\bullet\\ \vert \\ \bullet\end{smallmatrix}$.*

Proof. Let $(P, \leq)$ be a poset and suppose for contradiction P admits a reduction R isomorphic to $\cdot\ \cdot$ and a reduction S isomorphic to $\begin{smallmatrix}\bullet\\ \vert \\ \bullet\end{smallmatrix}$. Fix $x \in P$. Then there exist $y, z \in P$ with $y \notin [x]_R, z \notin [x]_S$. Note that members of distinct R classes must be incomparable while members of distinct S classes must be comparable, so $y \notin [x]_R$ implies y and x are incomparable which in turn implies $y \in [x]_S$. Similarly $z \in [x]_R$. Since $y \notin [x]_R$ which is a component it follows that $\leq^*(x, y) = \leq^*(z, y)$. Then since x and y are incomparable, so are z and y. But this entails $z \in [y]_S = [x]_S$, a contradiction. $\square$

Definition 6.9. With the preceeding lemma and theorem (6.2) we observe that for any finite, non-singleton poset $(P, \leq)$, exactly one of the following is true:

- P admits a reduction isomorphic to $\cdot\ \cdot$
- P admits a reduction isomorphic to $\begin{smallmatrix}\bullet\\ \vert \\ \bullet\end{smallmatrix}$
- P admits a reduction to an irreducible with order greater than 2

We say P is **parallel-canonical**, **series-canonical**, or **nonbinary-canonical** respectively.

Definition 6.10. The **cannonical reduction** of a poset is the common refinement of all its irreducible reductions.

The following lemma is used below in characterizing the canonical reduction.

Lemma 6.11. *If* $(P, \leq)$ *is a poset and* $\{\sim_\alpha\}_\alpha$ *are reduction relations with* $\sim := \bigcap_\alpha \sim_\alpha$*, then* $\forall x \in P, [x]_\sim = \bigcap_\alpha [x]_{\sim_\alpha}$.

Proof. Let $y \in P$. Then

$$\begin{aligned} & y \in [x]_\sim \\ \iff & y \sim x \\ \iff & y \sim_\alpha x, \forall \alpha \\ \iff & y \in [x]_{\sim_\alpha}, \forall \alpha \\ \iff & y \in \bigcap_\alpha [x]_{\sim_\alpha} \end{aligned}$$

□

6.2.1 Parallel Reductions

Definition 6.12. A non-trivial reduction of a poset is a **parallel reduction** if as a poset it is an antichain.

Remark. We observe that a poset being an antichain can be expressed as $\leq^*(x, y) = ||, \forall x \neq y$.

Lemma 6.13. *For a poset* $(P, \leq)$*, TFAE:*

1. *P is parallel-canonical (it admits a reduction isomorphic to* $\bullet\ \bullet$ *)*
2. *P can be partitioned into two subsets, with no element of the one comparable to an element of the other.*
3. *P admits a parallel reduction*

Proof. 2 is just a restatement of 1, and 1 clearly implies 3, so we prove 3 implies 2. Let R be a parallel reduction of P, and pick any X in R. Then $\forall Y \in R \setminus \{X\}$, since R is an antichain, X is incomparable to Y, so by (1.2.1) every element of X is incomparable to every element of Y, and so every element of X is incomparable to every element of $\bigcup R \setminus \{X\}$, and $\{X, \bigcup R \setminus \{X\}\}$ partitions P. □

Lemma 6.14. *The canonical reduction of a parallel-canonical poset is the finest parallel reduction.*

Proof. Let $(P, \leq)$ be a parallel-canonical poset with irreducible reductions R_α and let Q be the common refinement of $\{R_\alpha\}$. For all $[x]_Q \neq [y]_Q \in Q$, from (6.11) $[x]_Q = \bigcap_\alpha [x]_{R_\alpha}$ and $[y]_Q = \bigcap_\alpha [y]_{R_\alpha}$. Since $[x]_Q \neq [y]_Q$, $\exists \alpha, [x]_{R_\alpha} \neq [y]_{R_\alpha}$. R_α is a parallel reduction, so by (1.2.1)

$$|| = \leq^*_{R_\alpha}([x]_{R_\alpha}, [y]_{R_\alpha}) = \leq^*(x, y) = \leq^*_Q([x]_Q, [y]_Q)$$

That is Q is an antichain; a parallel reduction of P.

Now suppose R is a parallel reduction of P and let $X \in R$. Then as in the proof of (6.13), $\{X, \bigcup R \setminus X\}$ is a reduction of P isomorphic to $\cdot\ \cdot$. In particular, Q refines $\{X, \bigcup R \setminus X\}$, hence X is Q-compatible. Since this holds for all $X \in R$, Q refines R. □

Lemma 6.15. *A parallel reduction of a poset is the canonical reduction if and only if no component is parallel-canonical.*

Proof. Let $(P, \leq)$ be a poset and Q a parallel reduction of P. Note that P must be parallel-canonical by (6.13).

Suppose that no element of Q is parallel-canonical, and let $X \in Q$. If R is any parallel reduction, $\exists Z \in R, X \cap Z \neq \emptyset$. Since R is parallel, for any $Y \in R \setminus \{Z\}$, $\leq^*_R(Z, Y) = ||$. Thus for all $z \in Z, y \in \bigcup_{Y \in R \setminus \{Z\}} Y$, $\leq^*(z, y) = ||$. In particular for all $z \in X \cap Z, y \in X \cap \bigcup_{Y \in R \setminus \{R\}} Y$, $\leq^*(z, y) = ||$. But then we've partitioned X into a pair of subsets with incomparable elements; if both are non-empty from (6.13) X is parallel-canonical, a contradiction. So one must be empty and $X \cap Z \neq \emptyset$ so $X \cap \bigcup_{Y \in R \setminus \{R\}} Y = \emptyset$. Then $X = X \cap Z \implies X \subseteq Z$. So for all $X \in Q$, X is contained in an element of R; that is Q refines R.

Conversely suppose Q is the canonical reduction of P. Let $X \in Q$ and suppose S is a parallel reduction of X. We claim $R := (Q \setminus \{X\}) \cup S$ is a parallel reduction of P. R is a partition of P refining Q whose elements are all components of P: each element of S is a component of X, hence a component of P by (4.5) and each element of Q is a component of P. Thus R is a reduction of P by (4.2) and finer than Q by construction. Let $[x]_R \neq [y]_R \in R$ for $x, y \in P$. If $[x]_R, [y]_R \in S$, then $x, y \in X$, $[x]_R = [x]_S, [y]_R = [y]_S$ and S is parallel so

$$|| = \leq^*_S([x]_S, [y]_S) = \leq^*_R([x]_R, [y]_R)$$

If $[x]_R, [y]_R \in Q \setminus \{X\}$, $[x]_R = [x]_Q, [y]_R = [y]_Q$ and Q is parallel so

$$|| = \leq^*_Q([x]_Q, [y]_Q) = \leq^*_R([x]_R, [y]_R)$$

Otherwise, WLOG $[x]_R \in Q \setminus \{X\}$ and $[y]_R \in S$. Then $[y]_Q = X$ and

$$|| = \leq^*_Q([x]_Q, X) = \leq^*_Q([x]_Q, [y]_Q) = \leq^*(x, y) = \leq^*_R([x]_R, [y]_R)$$

Thus in all cases $\leq^*_R([x]_R, [y]_R) = ||$; i.e. R is an antichain. But then R is a parallel reduction refinining the canonical parallel reduction Q, so $R = Q$. In particular S is just the trivial reduction of X. The trivial reduction is by definition not parallel, so this is a contradiction. □

6.2.2 Series Reductions

Definition 6.16. A non-trivial reduction of a poset is a **series reduction** if as a poset it is a chain.

Notation. We use $X^{\uparrow R}$ to refer to the upper closure of $\{X\}$ in R, $\{Y \in R : X \leq_R Y\}$. Similarly for the lower closure $X^{\downarrow R}$.

Lemma 6.17. *For a poset $(P, \leq)$, TFAE:*

1. *P is series-canonical (it admits a reduction isomorphic to ⫯)*
2. *P can be partitioned into two subsets, with every element of one less than every element of the other.*
3. *P admits a series reduction*

Proof. 2 is just a restatement of 1, and 1 clearly implies 3, so we prove 3 implies 2.

Let R be a series reduction of P, and pick any X in R. Since R is non-singleton and total, at least one of $X^{\downarrow R} \setminus \{X\}$ or $X^{\uparrow R} \setminus \{X\}$ must be nonempty, so at least one of $\left\{\bigcup \left(X^{\downarrow R} \setminus \{X\}\right), \bigcup X^{\uparrow R}\right\}$ or $\left\{\bigcup X^{\downarrow R}, \bigcup \left(X^{\uparrow R} \setminus \{X\}\right)\right\}$ must be a partion of P into two non-empty subsets, and by construction all elements of one less than all elements of the other. $\square$

Lemma 6.18. *The canonical reduction of a series-canonical poset is the finest series reduction.*

Proof. Let $(P, \leq)$ be a series-canonical poset with irreducible reductions R_α and let Q be the common refinement of $\{R_\alpha\}$. For all $[x]_Q \neq [y]_Q \in Q$, from (6.11) $[x]_Q = \bigcap_\alpha [x]_{R_\alpha}$ and $[y]_Q = \bigcap_\alpha [y]_{R_\alpha}$. Since $[x]_Q \neq [y]_Q$, $\exists \alpha, [x]_{R_\alpha} \neq [y]_{R_\alpha}$. Then from (4.1.2)

$$\leq^*_{R_\alpha}([x]_{R_\alpha}, [y]_{R_\alpha}) = \leq^*(x, y) = \leq^*_Q([x]_Q, [y]_Q)$$

In particular, since R_α is a series reduction, $[x]_{R_\alpha}$ and $[y]_{R_\alpha}$ are comparable and hence so are $[x]_Q$ and $[y]_Q$. So Q is a chain; a series reduction of P.

Now suppose R is a series reduction of P and let $X \in R$. Let $S_1 := \left\{\bigcup \left(X^{\downarrow R} \setminus \{X\}\right), \bigcup X^{\uparrow R}\right\}$ and $S_2 := \left\{\bigcup X^{\downarrow R}, \bigcup \left(X^{\uparrow R} \setminus \{X\}\right)\right\}$. Then as in (6.17), either $X^{\downarrow R} \setminus \{X\}$ is empty meaning $\bigcup X^{\downarrow R} = P$ hence Q-compatible, or else S_1 is isomorphic to ⫯, in which case Q refines S_1 by construction and so $\bigcup X^{\downarrow R}$ is Q-compatible. So in all cases $\bigcup X^{\downarrow R}$ is Q-compatible, and similarly $\bigcup X^{\uparrow R}$ must be Q-compatible. But then for any $x \in X$, $x \in \bigcup X^{\downarrow R} \implies [x]_Q \subseteq \bigcup X^{\downarrow R}$ and $x \in \bigcup X^{\uparrow R} \implies [x]_Q \subseteq \bigcup X^{\uparrow R}$. Thus $[x]_Q \subseteq \bigcup X^{\downarrow R} \cap \bigcup X^{\uparrow R} = \bigcup\{X\} = X$, so X is Q-compatible. This holds for arbitrary $X \in R$, so Q refines R. $\square$

Lemma 6.19. *A series reduction of a poset is the canonical reduction if and only if no component is series-canonical.*

Proof. Let $(P, \leq)$ be a poset and Q a series reduction of P. Note that P must be series-canonical by (6.17).

Suppose that no element of Q is series-canonical, and let R be a series reduction. For any $X \in Q, \exists Z \in R, X \cap Z \neq \emptyset$. If $X \cap \bigcup \left(Z^{\downarrow R} \setminus \{Z\}\right) \neq \emptyset$ then $\left\{X \cap \bigcup \left(Z^{\downarrow R} \setminus \{Z\}\right), X \cap \bigcup Z^{\uparrow R}\right\}$ is a reduction of X isomorphic to $\mathbf{2}$, a contradiction. Similarly, if $X \cap \bigcup \left(Z^{\uparrow R} \setminus \{Z\}\right) \neq \emptyset$ then $\left\{X \cap \bigcup Z^{\downarrow R}, X \cap \bigcup \left(Z^{\uparrow R} \setminus \{Z\}\right)\right\}$ is a reduction of X isomorphic to $\mathbf{2}$. Thus $X \cap \bigcup \left(Z^{\downarrow R} \setminus \{Z\}\right) = \emptyset$ and $X \cap \bigcup \left(Z^{\uparrow R} \setminus \{Z\}\right) = \emptyset$; in other words $X \subseteq Z$. This holds for arbitrary $X \in Q$, so Q refines R.

Conversely suppose Q is the canonical reduction of P. Let $X \in Q$ and suppose S is a series reduction of X. We claim $R := (Q \setminus \{X\}) \cup S$ is a series reduction of P. R is a partition of P refining Q whose elements are all components of P: each element of S is a component of X, hence a component of P by (4.5), and each element of Q is a component of P. Thus R is a reduction of P by (4.2) and finer than Q by construction. Let $[x]_R \neq [y]_R \in R$. If $[x]_R, [y]_R \in S$ then $[x]_R = [x]_S$ and $[y]_R = [y]_S$ and since S is a series reduction these are comparable. If $[x]_R, [y]_R \in Q \setminus \{X\}$ then $[x]_R = [x]_Q$ and $[y]_R = [y]_Q$ and since Q is a series reduction these are comparable. Otherwise, WLOG $[x]_R \in Q \setminus \{X\}$ and $[y]_R \in S$. Then $[y]_Q = X \neq [x]_Q$, so from (4.1.2)

$$\leq^*_Q([x]_Q, [y]_Q) = \leq^*(x, y) = \leq^*_R([x]_R, [y]_R)$$

Since Q is a series reduction, $[x]_Q$ and $[y]_Q$ are comparable, so $[x]_R$ and $[y]_R$ must be comparable as well; i.e. R is a chain. But then R is a series reduction refinining the canonical series reduction Q, so $R = Q$. In particular S is just the trivial reduction of X. The trivial reduction is by definition not series, so this is a contradiction. □

6.3 Canonical composition

Definition 6.20. A poset is **canonical** if its canonical reduction is the identity reduction.

Theorem 6.21. *A poset is canonnical if and only if it is an antichain, a chain, or an irreducible of order greater than 2.*

Proof. If the identity reduction of a poset is the canonical reduction, it must be an antichain, a chain, or an irreducible of order greater than 2 from (6.9) and (6.7), (6.14), or (6.18); so the original poset must be as well (by identifying with its identity reduction).

Conversely, antichains, chains, and irreducibles of order greater than two are canonical:

- The identity reduction of an antichain is necessarily the finest parallel reduction, hence the canonical reduction from (6.14)
- The identity reduction of a chain is necessarily the finest series reduction, hence the canonical reduction from (6.18)
- An irreducible of order greater than two has such an identity reduction, which by (6.7) is the canonical reduction.

□

Definition 6.22. If $(P_\alpha, \leq_\alpha)$ are disjoint posets and $\leq$ is a partial order on $\{P_\alpha\}_\alpha$, the composition $\leq [\leq_\alpha]_\alpha$ is **canonical** if $\{P_\alpha\}_\alpha$ is non-singleton and is the canonical reduction of $(\bigcup_\alpha P_\alpha, \leq [\leq_\alpha]_\alpha)$.

We explicitly exclude composition through a singleton from being canonical.

Lemma 6.23. *For disjoint posets $(P_\alpha, \leq_\alpha)$ and partial order $\leq$ on $\{P_\alpha\}_\alpha$, the composition $\leq [\leq_\alpha]_\alpha$ is canonical if and only if $(\{P_\alpha\}_\alpha, \leq)$ is a canonical poset and*

- *If it is an antichain, no P_α is parallel-canonical*
- *If it is a chain, no P_α is series-canonical*

Proof. ($\Longrightarrow$) As a canonical reduction $(\{P_\alpha\}_\alpha, \leq)$ is an antichain, chain or irreducible of order greater than two, and thus a canonical poset by (6.21). Further, if it is an antichain then by (6.15) no P_α is parallel-canonical, and if it is a chain then by (6.19) no P_α is series-canonical.

($\Longleftarrow$) As a canonical poset $(\{P_\alpha\}_\alpha, \leq)$ is an antichain, a chain or an irreducible of order greater than two by (6.21), and is by definition a reduction of $(\bigcup_\alpha P_\alpha, \leq [\leq_\alpha]_\alpha)$.

- If it is an antichain then by assumption and (6.15) it is the canonical reduction.
- If it is a chain then by assumption and (6.19) it is the canonical reduction.
- If it is an irreducible of degree greater than two then by (6.7) it must be the canonical reduction.

In all cases this means by definition the composition $\leq [\leq_\alpha]_\alpha$ is canonical. □

Definition 6.24. A **composistion tree** is a triple $(\mathcal{T}, P, \leq_P)$ with $\leq_P$ a partial order on P and $\mathcal{T}$ a set of composition trees, defined recursively:

- For a singleton poset $(\{p\}, \leq_p)$, $(\{\}, \{p\}, \leq_p)$ is a composition tree.
- If $\{T_\alpha\}_\alpha = \{(\mathcal{T}_\alpha, P_\alpha, \leq_\alpha)\}_\alpha$ are composition trees with $\{P_\alpha\}$ pairwise disjoint, and $\leq$ is a partial order on $\{P_\alpha\}_\alpha$, then

  $$\left(\{T_\alpha\}_\alpha, \bigcup_\alpha P_\alpha, \leq [\leq_\alpha]_\alpha\right)$$

  is a composition tree.

  A **canonical composition tree** is a composition tree where every composition is canonical.

Remark 6.25. From (6.23) we observe that canonical composition trees are composition trees where each interior node is a canonical poset, with no antichain children of antichain nodes and no chain children of chain nodes.

Theorem 6.26. *Every finite poset can be obtained via a unique canonical composition tree.*

Proof. We prove this by induction on the size of the poset.

By definition every singleton has an associated canonical composition tree with no subtrees, and since canonical compositions are non-singleton, it must be unique.

Now let $n \in \mathbb{N}$ and suppose $\forall m \in \mathbb{N}, m < n$ every poset of size m is obtained from a unique canonical composition tree. Let $(P, \leq_P)$ be a poset of order n and let Q be its canonical reduction. Then for each $X \in Q$, since Q is non-singleton, $|X| < n$. Letting $\leq_X := \leq_P|_X$, from the induction hypothesis $(X, \leq_X)$ is obtained from a unique canonical composition tree $T_X = (\mathcal{T}_X, X, \leq_X)$. Then

- The elements of Q are pairwise disjoint,
- $P = \bigcup_{X \in Q} X$,
- by definition (2.3) $\leq_P = \leq_Q [\leq_X]_{X \in Q}$,
- and as Q is the canonical reduction of P, that composition is canonical.

It follows that

$$T_P = \left(\{T_X\}_{X \in Q}, \bigcup_{X \in Q} X, \leq_Q [\leq_X]_{X \in Q} \right) = \Big(\{T_X\}_{X \in Q}, P, \leq_P \Big)$$

is a canonical composition tree yielding P.

Further this must be unique: Suppose T'_P were another canonical composition tree yielding P. Since P is non-singleton, T'_P must be of the form $T'_P = (\{T_\alpha\}_\alpha, P, \leq_P)$ for some canonical composition trees $\{T_\alpha\}_\alpha = \{(\mathcal{T}_\alpha, P_\alpha, \leq_\alpha)\}_\alpha$ and a partial order $\leq_{Q'}$ on $Q' := \{P_\alpha\}_\alpha$, where $\bigcup_\alpha P_\alpha = P$, $\leq_{Q'} [\leq_\alpha]_\alpha = \leq_P$, and that composition must be canonical. But then both Q and Q' must be the canonical reduction for P, which is unique, so it follows that $Q = Q'$. Further each $X \in Q$ must equal P_α for some α, and since the canonical composition trees for each X are unique, it follows that $T_X = T_\alpha$. Then

$$T_P = \Big(\{T_X\}_{X \in Q}, P, \leq_P \Big) = (\{T_\alpha\}_\alpha, P, \leq_P) = T'_P$$

completing the proof. □

Together remark (6.25) and theorem (6.26) show that every finite poset can be obtained in a unique way from the class of canonical posets, which consist of the antichains, chains, and irreducibles of order greater than 2. Thus in principle it should be possible to classify all finite posets by classifying the irreducibles of order greater than 2. In particular, they should be relatable by combinatorial species:

Let **Po**, **P**, $\hat{\mathbf{P}}$, **S**, $\hat{\mathbf{S}}$, **N**, and $\hat{\mathbf{N}}$ denote the species of posets, anti-chains, parallel-canonical posets, chains, series-canonical posets, irreducible posets of order greater than two, and non-binary canonical posets respectively. Then they are related in the following ways:

- $\mathbf{Po} = \mathbf{X} + \hat{\mathbf{P}} + \hat{\mathbf{S}} + \hat{\mathbf{N}}$
- $\hat{\mathbf{P}} = \mathbf{P} \circ (\mathbf{Po} - \hat{\mathbf{P}})$
- $\hat{\mathbf{S}} = \mathbf{S} \circ (\mathbf{Po} - \hat{\mathbf{S}})$
- $\hat{\mathbf{N}} = \mathbf{N} \circ \mathbf{Po}$

where $\mathbf{X}$ is the singleton species. Given that $\mathbf{P}$ and $\mathbf{S}$ are known to us, this should reduce to an expression relating $\mathbf{Po}$ to $\mathbf{N}$.